Vector Product or Cross Product

If $\vec{a}$and $\vec{b}$ are two vectors and θ is the angle between them , then the vector product of these two vectors denoted by $\vec{a} \times \vec{b}$ is defined as

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}| \sin\theta \hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$and $\vec{b}$.

As shown in figure-21 the direction of $\vec{a} \times \vec{b}$ is always perpendicular to both $\vec{a}$and$\vec{b}$.

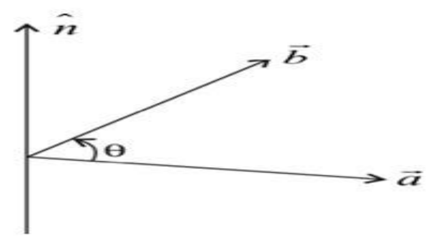

Fig-22

Properties of cross product

i) Vector product is not commutative $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

ii) For any two vectors $\vec{a}$and $\vec{b}$, $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

iii) For any scalar m, $m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

iii) Distributive $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

iv) Vector product of two parallel or collinear vectors is zero.

$\vec{a} \times \vec{a} = \vec{0}$ and if $\vec{a} \parallel \vec{b}$ then $\vec{a} \times \vec{b} = \vec{0}$ { as θ = 0 or 180º => sin θ = 0 }

Using this property we have,

$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}$ = $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}$ = $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}$ = $\vec{0}$

v) Vector product of orthonormal unit vectors form a right handed system.

As shown in figure- 23 the three mutually perpendicular unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ form a right handed system , i.e. $\hat{\imath} \times \hat{\jmath} = \hat{k} = -(\hat{\jmath} \times \hat{\imath})$ (as θ =90 , then sin θ = 1)

$\hat{\jmath} \times \hat{k} = \hat{\imath} = -(\hat{k} \times \hat{\jmath})$

$\hat{k} \times \hat{\imath} = \hat{\jmath} = -(\hat{\imath} \times \hat{k})$

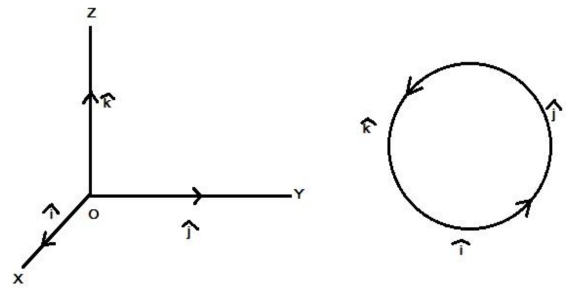

Fig-23

Unit vector perpendicular to two vectors:- Unit vector perpendicular to two given vectors $\vec{a}$and $\vec{b}$ is given by $\hat{\boldsymbol{n}} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

<u>Angle between two vectors</u>

Let θ be the angle between $\vec{a}$and $\vec{b}$. Then $\vec{a} \times \vec{b}$ = (I $\vec{a}$ I. I $\vec{b}$ I sin θ) $\hat{n}$.

Taking modulus of both sides we have,

$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}| \sin\theta$

⇨ $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|.|\vec{b}|}$

Hence $\boldsymbol{\theta} = \boldsymbol{sin^{-1}} \left\{\frac{|\vec{a} \times \vec{b}|}{|\vec{a}|.|\vec{b}|}\right\}$

Geometrical Interpretation of vector product or cross product

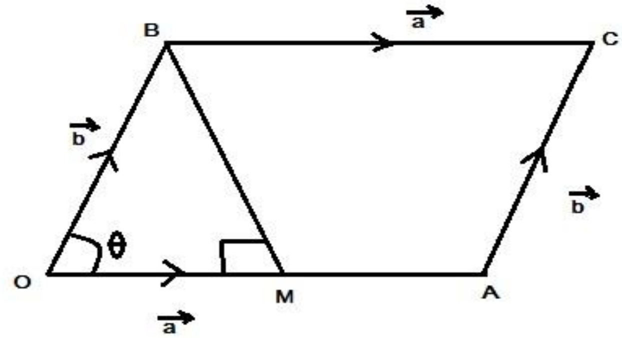

Fig-24

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Then $\vec{a} \times \vec{b}$ = (I $\vec{a}$ I. I $\vec{b}$ I sin θ) $\hat{n}$

= (I $\vec{a}$ I).(I $\vec{b}$ I sin θ)$\hat{n}$

From fig-24 below it is clear that

BM = OB sin θ = I $\vec{b}$ I sin θ = I $\vec{a}$ I IBMI $\hat{n}$

{ as sin θ = BM/OB &$\overrightarrow{OB} = \vec{b}$}

Now I $\vec{a} \times \vec{b}$I = I $\vec{a}$ I IBMI I $\hat{n}$ I = OA.

BM = Area of the parallelogram with side $\vec{a}$ and $\vec{b}$.

Therefore the magnitude of cross product of two vectors is equal to area of the parallelogram formed by these vectors as two adjacent sides.

From this it can be concluded that area of $\triangle$ ABC = ½ I$\overrightarrow{AB}$. $\overrightarrow{AC}$I

Application of cross product

1. Moment of a force about a point ($\vec{M}$) :-Let O be any point and Let $\vec{r}$ be the position vector w.r.t. O of any point 'P' on the line of action of the force $\vec{F}$, then the moment or torque of the force F about origin 'O' is given by

$\vec{M} = \vec{r} \times \vec{F}$

2. If $\vec{a}$and $\vec{b}$ represent two adjacent sides of a triangle then the area of the triangle is given by

$\Delta = \frac{1}{2}|\vec{a} \times \vec{b}|$ Sq. unit

3. If $\vec{a}$and $\vec{b}$ represent two adjacent sides of a parallelogram then area of the parallelogram is given by $\Delta = |\vec{a} \times \vec{b}|$ Sq. unit

4. If $\vec{a}$and $\vec{b}$ represent two diagonals of a parallelogram then area of the parallelogram is given by Δ $= \frac{1}{2}|\vec{a} \times \vec{b}|$ Sq. unit

Vector product in component form : -

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$.

$\vec{a} \times \vec{b} = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$

$= a_1b_1(\hat{\imath} \times \hat{\imath}) + a_1b_2(\hat{\imath} \times \hat{\jmath}) + a_1b_3(\hat{\imath} \times \hat{k}) + a_2b_1(\hat{\jmath} \times \hat{\imath}) + a_2b_2(\hat{\jmath} \times \hat{\jmath}) + a_2b_3(\hat{\jmath} \times \hat{k})$

$+ a_3b_1(\hat{k} \times \hat{\imath}) + a_3b_2(\hat{k} \times \hat{\jmath}) + a_3b_3(\hat{k} \times \hat{k})$

{ using properties $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$, $\hat{\imath} \times \hat{\jmath} = \hat{k} = -(\hat{\jmath} \times \hat{\imath})$, $\hat{\jmath} \times \hat{k} = \hat{\imath} = -(\hat{k} \times \hat{\jmath})$ and

$\hat{k} \times \hat{\imath} = \hat{\jmath} = -(\hat{\imath} \times \hat{k})$ }

$= (a_2b_3 - a_3b_2)\,\hat{\imath} + (a_3b_1 - a_1b_3)\,\hat{\jmath} + (a_1b_2 - a_2b_1)\,\hat{k}$

$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ i.e. $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Condition of Co-planarity

If three vectors $\vec{a}, \vec{b}$and $\vec{c}$ lies on the same plane then the perpendicular to $\vec{a}$ *and* $\vec{b}$ must be perpendicular to $\vec{c}$.

In particular $(\vec{a} \times \vec{b}) \perp \vec{c}$ => $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

In component form if $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$,$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

Then $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

- ⇨ $(a_2b_3 - a_3b_2)\,c_1 + (a_3b_1 - a_1b_3)\,c_2 + (a_1b_2 - a_2b_1)\,c_3 = 0$
- ⇨ $\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$ (interchanging rows two times R_1 *and* R_2 *, then* R_2 *and* R_3)

$\Rightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

Example:- 17

If $\vec{a} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{b} = -\hat{\imath} + 3\hat{k}$ then find $|\vec{a} \times \vec{b}|$

Ans: - We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$

$= \{(3\times3)-(0\times(-2))\}\hat{\imath} - \{(1\times3)-((-1)\times(-2))\}\hat{\jmath} + \{(1\times0)-((-1)\times3)\}\hat{k}$

$= 9\hat{\imath} - \hat{\jmath} + 3\hat{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{81+1+9} = \sqrt{91}$ (Ans)

Example:-18 Determine the area of the parallelogram whose adjacent sides are the vectors

$\vec{a} = \widehat{2\imath}$ and $\vec{b} = 3\hat{\jmath}$. (2013-W)

Ans:- Area of the parallelogram with adjacent sides given by $\vec{a}$ and $\vec{b}$ is given by

area = $|\vec{a} \times \vec{b}| = |\widehat{2\imath} \times 3\hat{\jmath}| = |6\hat{k}| = 6$ sq units (Ans)

Example:-19 Find a unit vector perpendicular to both the vectors $\vec{a} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = 3\hat{\imath} - \hat{\jmath} + 3\hat{k}$.

Ans: - (2015-W and 2017-S)

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by

$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$……………….(1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 3 & -1 & 3 \end{vmatrix}$

$= (3-1)\hat{\imath} - (6+3)\hat{\jmath} + (-2-3)\hat{k}$

$= 2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}$-----------------(2)

From (1) and (2) we have,

$$\hat{n} = \frac{2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}}{|\,2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}\,|} = \frac{2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}}{\sqrt{2^2+(-9)^2+(-5)^2}} = \frac{2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}}{\sqrt{110}}$$

$$= \frac{2}{\sqrt{110}}\hat{\imath} - \frac{9}{\sqrt{110}}\hat{\jmath} - \frac{5}{\sqrt{110}}\hat{k} \text{ (ans)}$$

Example:-20 If $\vec{a} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{b} = 3\,\hat{\imath} + 4\hat{\jmath} - \hat{k}$, then find the sine of the angle between these vectors. (2016-w)

Ans :- We know that $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|.|\vec{b}|}$................(1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$

$= (1-4)\,\hat{\imath} - (-2 - 3)\,\hat{\jmath} + (8 + 3)\,\hat{k} = -3\,\hat{\imath} + 5\,\hat{\jmath} + 11\,\hat{k}$

Hence $|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + 5^2 + 11^2} = \sqrt{9 + 25 + 121} = \sqrt{155}$ ....(2)

Again $|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$ ....(3)

and $|\vec{b}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{9 + 16 + 1} = \sqrt{26}$.............................(4)

From equation (1),(2),(3) and (4) we have,

$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|.|\vec{b}|} = \frac{\sqrt{155}}{\sqrt{6}\sqrt{26}} = \frac{\sqrt{155}}{\sqrt{156}}$ (Ans)

Q-21 Calculate the area of the triangle ABC (by vector method) where A(1,1,2), B(2,2,3) and C(3,-1,-1). (2013-W)

Solution: - Let the position vector of the vertices A,B and C is given by $\vec{a}$,$\vec{b}$ and $\vec{c}$ respectively.

Then $\vec{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\vec{b} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\vec{c} = 3\hat{\imath} - \hat{\jmath} - \hat{k}$

Now $\overrightarrow{AB} = Position\ vector\ of\ B - Position\ vector\ of\ A$

$= 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k} - (\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$= (2-1)\hat{\imath} + (2-1)\hat{\jmath} + (3-2)\hat{k}$

$= \hat{\imath} + \hat{\jmath} + \hat{k}$

Similarly $\overrightarrow{AC} = Position\ vector\ of\ C - Position\ vector\ of\ A$

$= 3\hat{\imath} - \hat{\jmath} - \hat{k} - (\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$= (3-1)\hat{\imath} + (-1-1)\hat{\jmath} + (-1-2)\hat{k}$

$= 2\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$

Now $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -2 & -3 \end{vmatrix}$

$= (-3 + 2)\,\hat{\imath} - (-3 - 2)\,\hat{\jmath} + (-2 - 2)\,\hat{k} = -\hat{\imath} + 5\hat{\jmath} - 4\hat{k}$

Hence area of the triangle is given by

$\Delta = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{(-1)^2 + 5^2 + (-4)^2}$

$= \frac{1}{2}\sqrt{1 + 25 + 16} = \frac{1}{2}\sqrt{42}$ sq units. (Ans)

Example:-22 Find the area of a parallelogram whose diagonals are determined by the vectors

$\vec{a} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$. (2014-W, 2017-W)

Ans: - Area of the parallelogram with diagonals $\vec{a}\ and\ \vec{b}$ are given by

$$\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

$= (4 - 6)\,\hat{\imath} - (12 + 2)\,\hat{\jmath} + (-9 - 1)\,\hat{k} = -2\,\hat{\imath} - 14\,\hat{\jmath} - 10\,\hat{k}$

Now area $\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{(-2)^2 + (-14)^2 + (-10)^2}$

$= \frac{1}{2}\sqrt{4 + 196 + 100} = \frac{\sqrt{300}}{2} = \frac{10\sqrt{3}}{2} = 5\sqrt{3}$ sq unit. (ans)

Example:-23 For any vector $\vec{a}$ *and* $\vec{b}$, prove that $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a}\,.\vec{b})^2$ where a and b are magnitude of $\vec{a}$ *and* $\vec{b}$ respectively.

Proof: - $(\vec{a} \times \vec{b})^2 = (|\vec{a}|.|\vec{b}|\sin\theta\hat{n})^2$

$= (ab\sin\theta\hat{n})^2 = a^2b^2sin^2\theta$ (As $(\hat{n})^2 = (|\hat{n}|)^2 = 1^2 = 1$)

$= a^2b^2(1 - cos^2\theta) = a^2b^2 - a^2b^2cos^2\theta$

$= a^2b^2 - (abcos\theta)^2 = a^2b^2 - (\vec{a}\,.\vec{b})^2$ (Proved)

Example:-24 In a ΔABC , prove by vector method that $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$,

where BC = a, CA = b and AB =c. (2017-S)

Proof:- As shown in figure- 25 ABC is a triangle having, $\vec{a} = \overrightarrow{BC}, \vec{b} = \overrightarrow{CA}$ and $\vec{c} = \overrightarrow{AB}$.

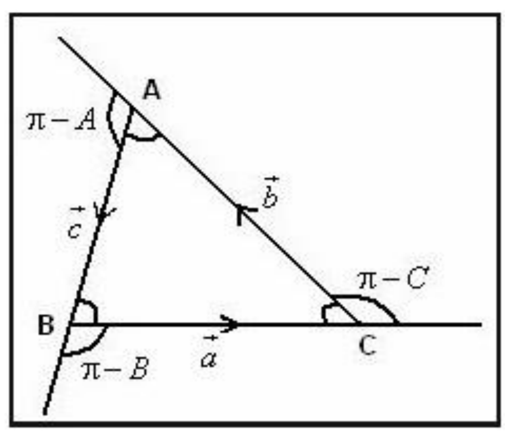

Fig-25

From triangle law of vector we know that ,

$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

⇨ $\vec{a} + \vec{b} = -\vec{c}$

⇨ $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ ……….(1)

(taking cross product of both sides with $\vec{a}$ we get)

⇨ $\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

⇨ $(\vec{a} \times \vec{a}) + (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) = \vec{0}$

⇨ $\vec{0} + (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) = \vec{0}$

⇨ $(\vec{a} \times \vec{b}) = -(\vec{a} \times \vec{c})$

⇨ $(\vec{a} \times \vec{b}) = (\vec{c} \times \vec{a})$ ---------------------(2)

Similarly taking cross product with $\vec{b}$ both sides of (1) we have,

⇨ $(\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c})$ ------------------(3)

From (2) and (3) , $(\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{a})$

⇨ $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

⇨ $ab\sin(\pi - C) = bc\sin(\pi - A) = ca\sin(\pi - B)$

As from fig-25 it is clear that angle between $\vec{a}$ and $\vec{b}$ is $-C$, $\vec{b}$ and $\vec{c}$ is $\pi - A$and $\vec{c}$ and $\vec{a}$ is $\pi - B$.

Dividing above equation by abc we have,

⇨ $\frac{ab\sin(\pi - C)}{abc} = \frac{bc\sin(\pi - A)}{abc} = \frac{ca\sin(\pi - B)}{abc}$

⇨ $\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

Hence $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$ (Proved).

Example:-25 What inference can you draw when $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = \vec{0}$

Ans: - Given $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = \vec{0}$

⇨ { Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ II $\vec{b}$ } and { $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$ }

⇨ As $\vec{a}$ II $\vec{b}$ and $\vec{a} \perp \vec{b}$ cannot be hold simultaneously so $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

Hence either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

Example:-26 If $|\vec{a}| = 2$ and $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, then find $\vec{a}.\vec{b}$.

Ans: - Given $|\vec{a} \times \vec{b}| = 8$

⇨ $|\vec{a}|\,|\vec{b}|\sin\theta = 8$

⇨ $2 \times 5 \sin\theta = 8$

⇨ $\sin\theta = \frac{8}{10} = \frac{4}{5}$

∴ $\cos\theta = \sqrt{1 - sin^2\theta} = \sqrt{1 - (\frac{4}{5})^2} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

Hence $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = 2 \times 5 \times \frac{3}{5} = 6$ (Ans)

Example:-27 Show that the vectors $\hat{\imath} - 3\,\hat{\jmath} + 4\hat{k}$, $2\hat{\imath} - \hat{\jmath} + 2\hat{k}$, and $4\hat{\imath} - 7\,\hat{\jmath} + 10\hat{k}$ are co-planar.(2017-S)

Ans: - Now let us find the following determinant ,

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & 2 \\ 4 & -7 & 10 \end{vmatrix} = 1(-10+14) – (-3)\,(20-8) + 4(-14+4) = 4 + 36 - 40 = 0$$

Hence the three given vectors are co-planar.